

# PRINCIPAL TORUS BUNDLES OF LORENTZIAN S-MANIFOLDS AND THE $\varphi$ -NULL OSSERMAN CONDITION

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**ABSTRACT.** The main result we give in this brief note relates, under suitable hypotheses, the  $\varphi$ -null Osserman, the null Osserman and the classical Osserman conditions to each other, via semi-Riemannian submersions as projection maps of principal torus bundles arising from a Lorentzian  $\mathcal{S}$ -manifold.

## 1. INTRODUCTION

The Jacobi operator is, for several reasons, one of the most interesting objects induced by the curvature operator.

On a (semi-)Riemannian manifold  $(M, g)$ , let us consider the unit spacelike  $S^+(M)$  (resp. timelike  $S^-(M)$ ) sphere bundle with fiber

$$S_p^\pm(M) = \{z \in T_p M \mid g_p(z, z) = \pm 1\},$$

and put  $S(M) = \bigcup_{p \in M} S_p^+(M) \cup S_p^-(M)$ .

For any  $z \in S_p(M)$ ,  $p \in M$ , the *Jacobi operator with respect to  $z$*  is the endomorphism  $R_z: z^\perp \rightarrow z^\perp$  such that  $R_z(\cdot) = R_p(\cdot, z)z$  ([20]), where  $R$  is the  $(1, 3)$ -type curvature tensor on  $(M, g)$ .

The Jacobi operator is obviously self-adjoint, hence a great deal of study has been carried out about the behaviour of its eigenvalues in the Riemannian case since R. Osserman proposed his Conjecture in [33] (see also [32]). Indeed, one easily sees that Riemannian space-forms are characterized by having Jacobi operators with exactly one constant eigenvalue corresponding to the sectional curvature. Those Riemannian manifolds whose Jacobi operators have eigenvalues independent both of the vector  $z \in S_p(M)$  and of the point  $p \in M$  are the *Osserman manifolds*. Any locally flat or locally rank-one symmetric space is an Osserman manifold, whilst the converse statement is known as the Osserman Conjecture. Several authors have dealt with this Conjecture, providing positive answers in many cases ([12], [13], [14], [28], [29], [30]).

One gets a different situation when considering the indefinite setting, where a semi-Riemannian manifold  $(M, g)$  is said to be *spacelike* (resp. *timelike*) *Osserman*, if the characteristic polynomial of  $R_z$  is independent of both  $z \in S_p^+(M)$  (resp.  $z \in S_p^-(M)$ ) and  $p \in M$ . It is known that  $(M, g)$  being spacelike Osserman is equivalent to  $(M, g)$  being timelike Osserman ([19], [20]), but several counterexamples to the Osserman Conjecture were found (see for example [5], [6], [21]) for non-Lorentzian semi-Riemannian manifolds.

Finally, in the Lorentzian setting a complete solution for the Osserman Conjecture was provided in a sequence of works by E. García-Río, D.N. Kupeli and M.E. Vázquez-Abal ([17], [18]), together with N. Blazić, N. Bokan and P. Gilkey

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([4]). They proved that a Lorentzian manifold is Osserman if and only if it has constant sectional curvature (see also [20]).

It was defined a very fruitful new Osserman-related condition for Lorentzian manifolds in [18]. There, the authors introduced the Jacobi operator  $\bar{R}_u$  with respect to a null (lightlike) vector  $u$ , and then they studied the so-called *null Osserman conditions* with respect to a unit timelike vector (see also [20]).

Here, we are concerned with an Osserman-related condition derived by the null Osserman condition, which is known as the  $\varphi$ -null Osserman condition, introduced and studied by the first author in [7] for manifolds carrying Lorentzian globally framed  $f$ -structures. This condition appears to be a natural generalization of the null Osserman condition, to which it reduces when considering Lorentzian almost contact structures. This was motivated by the following considerations: although any Lorentzian Sasaki manifold  $(M, \varphi, \xi, \eta, g)$  with constant  $\varphi$ -sectional curvature is globally null Osserman with respect to the timelike vector field  $\xi$ , there is no similar result when we consider Lorentzian  $\mathcal{S}$ -manifolds, which generalize Lorentzian Sasaki ones, and moreover, as we proved in [8], no Lorentzian  $\mathcal{S}$ -manifold can be neither null Osserman, nor Osserman. Further basic properties of such manifolds are studied in [7] and developed in [8]. We refer the reader to both works for more details about the  $\varphi$ -null Osserman condition, whilst the general reference for the whole Osserman framework is [20].

In this short note, we deal with the study of some relationships among the above three Osserman-related notions, providing a few results of equivalence, obtained by considering a natural structure of principal torus bundle arising from a Lorentzian  $\mathcal{S}$ -manifold, which involves semi-Riemannian submersions.

Indeed, from [3], a strong link between  $f$ -structures and Riemannian submersions is well-known. Namely, any compact and connected manifold endowed with a regular and normal  $g.f.f$ -structure is the total space of a torus principal bundle over a complex manifold, which, under suitable hypotheses, can be a Kähler manifold. Moreover, as also proved in [3], a compact, connected and regular Riemannian  $\mathcal{S}$ -manifold  $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ , with each  $\xi_\alpha$  regular, projects itself onto a compact Kähler manifold and onto a compact and regular Sasakian manifold. These results have been extended to the semi-Riemannian case by the first author, together with A.M. Pastore, who in [10] proved that a compact, connected and regular indefinite (in particular, Lorentzian)  $\mathcal{S}$ -manifold  $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$  projects itself onto a compact (indefinite) Kähler manifold and onto a compact and regular indefinite (Lorentzian) Sasakian manifold, via semi-Riemannian submersions.

Based on the above, after recalling, in Section 2, some basic features of (almost)  $\mathcal{S}$ -manifolds, in Section 4 we carry on an investigation on the possibilities of projectability of the  $\varphi$ -null Osserman conditions via semi-Riemannian submersions with a Lorentzian  $\mathcal{S}$ -manifold as total space, and either a Lorentzian Sasakian manifold or a Kähler manifold as base space. Using some properties established in [8], which we briefly recall in Section 3, together with a few properties of semi-Riemannian submersions, and under an additional assumption on the eigenvectors of the Jacobi operators, we obtain equivalence results relating the  $\varphi$ -null Osserman condition with the classical and the null Osserman condition in the framework of principal torus bundles constructed on a given Lorentzian  $\mathcal{S}$ -manifold.

In what follows, all smooth manifolds are supposed to be connected, and all tensor fields and maps are assumed to be smooth. Moreover, according to [24], for the Riemannian curvature tensor of a semi-Riemannian manifold  $(M, g)$  we use the definition  $R(X, Y, Z, W) = g(R(Z, W)Y, X) = g([\nabla_Z, \nabla_W]Y, X)$  for any vector fields  $X, Y, Z, W$  on  $M$ .

Finally, for any  $p \in M$  and any linearly independent vectors  $x, y \in T_p M$  spanning a non-degenerate plane  $\pi = \text{span}(x, y)$ , that is  $g_p(x, x)g_p(y, y) - g_p(x, y)^2 \neq 0$ , the sectional curvature of  $(M, g)$  at  $p$  with respect to  $\pi$  is, by definition, the real number

$$k_p(\pi) = k_p(x, y) = \frac{R_p(x, y, x, y)}{\Delta(\pi)},$$

where  $\Delta(\pi) = g_p(x, x)g_p(y, y) - g_p(x, y)^2$ .

## 2. PRELIMINARIES

Let us recall some basic definitions and facts about (almost)  $\mathcal{S}$ -manifolds needed in the rest of the paper.

Framed  $f$ -manifolds were originally considered by H. Nakagawa in [26] and [27], based on the notion of  $f$ -structure, which was firstly introduced in 1963 by K. Yano ([36]) as a generalization of both (almost) contact and (almost) complex structures. Such structures were later studied and developed by S.I. Goldberg and K. Yano (see, for example, [22], [23]) and, in the subsequent years, by several authors ([1], [3], [11], [25], [35]).

A *globally framed  $f$ -structure* (briefly  *$g.f.f$ -structure*) on a manifold  $M$  is a non-vanishing  $(1, 1)$ -type tensor field  $\varphi$  on  $M$  of constant rank satisfying the following conditions:  $\varphi^3 + \varphi = 0$ , and the subbundle  $\ker(\varphi)$  is parallelizable. This is equivalent to the existence of  $s$  linearly independent vector fields  $\xi_\alpha$  and 1-forms  $\eta^\alpha$  ( $\alpha \in \{1, \dots, s\}$ ),  $s$  being the dimension of  $\ker(\varphi)$  at any point  $p \in M$ , such that

$$(2.1) \quad \varphi^2 = -I + \eta^\alpha \otimes \xi_\alpha \quad \text{and} \quad \eta^\alpha(\xi_\beta) = \delta_\beta^\alpha.$$

Each  $\xi_\alpha$  is said to be a *characteristic vector field* of the structure, and a manifold  $M$  carrying a  $g.f.f$ -structure is denoted by  $(M, \varphi, \xi_\alpha, \eta^\alpha)$ , and called a  *$g.f.f$ -manifold*. When  $s = 1$  (resp.:  $s = 0$ ), we have an almost contact (resp.: almost complex) structure. From (2.1) one easily has  $\varphi\xi_\alpha = 0$  and  $\eta^\alpha \circ \varphi = 0$ , for any  $\alpha \in \{1, \dots, s\}$ . Furthermore,  $\text{Im}(\varphi)$  is a distribution on  $M$  of even rank  $r = 2n$  on which  $\varphi$  acts as an almost complex tensor field, and one has the splitting  $TM = \text{Im}(\varphi) \oplus \ker(\varphi)$ , hence  $\dim(M) = 2n + s$ . A  $g.f.f$ -manifold is said to be *normal* if the  $(1, 2)$ -type tensor field  $N = [\varphi, \varphi] + 2d\eta^\alpha \otimes \xi_\alpha$  vanishes.

In [9], the authors study the properties of a  $g.f.f$ -manifold  $(M, \varphi, \xi_\alpha, \eta^\alpha)$  endowed with a compatible indefinite metric, that is a semi-Riemannian metric  $g$  verifying

$$(2.2) \quad g(\varphi X, \varphi Y) = g(X, Y) - \sum_{\alpha=1}^s \varepsilon_\alpha \eta^\alpha(X) \eta^\alpha(Y),$$

for all  $X, Y \in \Gamma(TM)$ , where  $\varepsilon_\alpha = g(\xi_\alpha, \xi_\alpha) = \pm 1$ . Such a manifold is said to be an *indefinite metric  $g.f.f$ -manifold* and denoted by  $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ . From (2.2) one also has  $g(X, \xi_\alpha) = \varepsilon_\alpha \eta^\alpha(X)$  and  $g(X, \varphi Y) = -g(\varphi X, Y)$ , for any  $X, Y \in \Gamma(TM)$ , and the splitting  $TM = \text{Im}(\varphi) \oplus \ker(\varphi)$  becomes orthogonal.

The *fundamental 2-form*  $\Phi$  of an indefinite metric  $g.f.f$ -manifold  $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$  is defined by  $\Phi(X, Y) = g(X, \varphi Y)$ . If  $\Phi = d\eta^\alpha$ , for any  $\alpha \in \{1, \dots, s\}$ , the manifold is said to be an *indefinite almost  $\mathcal{S}$ -manifold*. Finally, a normal indefinite almost  $\mathcal{S}$ -manifold is, by definition, an *indefinite  $\mathcal{S}$ -manifold*. Such a manifold is characterized by the identity  $(\nabla_X \varphi)Y = g(\varphi X, \varphi Y)\bar{\xi} + \bar{\eta}(Y)\varphi^2 X$ , where  $\bar{\xi} = \sum_{\alpha=1}^s \xi_\alpha$  and  $\bar{\eta} = \sum_{\alpha=1}^s \varepsilon_\alpha \eta^\alpha$ . It follows that  $\nabla_X \xi_\alpha = -\varepsilon_\alpha \varphi X$  and  $\nabla_{\xi_\alpha} \xi_\beta = 0$ , for any  $\alpha, \beta \in \{1, \dots, s\}$ , and each  $\xi_\alpha$  is a Killing vector field.

For more details on (almost)  $\mathcal{S}$ -manifolds the reader is referred to [15] in the Riemannian case, and to [9] for the indefinite case.

### 3. LORENTZIAN $\mathcal{S}$ -MANIFOLDS AND THE $\varphi$ -NULL OSSERMAN CONDITION.

The notion of  $\varphi$ -null Osserman condition is derived from that of null Osserman, which we briefly recall here, following [18] and [20].

Let  $(M, g)$  be a Lorentzian manifold and  $p \in M$ . If  $u \in T_p M$  is a lightlike (or null) vector, that is  $u \neq 0$  and  $g_p(u, u) = 0$ , then  $\text{span}(u) \subset u^\perp$ . We can endow the quotient space  $\bar{u}^\perp = u^\perp / \text{span}(u)$ , whose canonical projection is  $\pi: u^\perp \rightarrow \bar{u}^\perp$ , with a positive definite inner product  $\bar{g}$  defined by  $\bar{g}(\bar{x}, \bar{y}) = g_p(x, y)$ , where  $\pi(x) = \bar{x}$  and  $\pi(y) = \bar{y}$ , obtaining the Euclidean vector space  $(\bar{u}^\perp, \bar{g})$ .

The *Jacobi operator with respect to  $\bar{u}$*  is the endomorphism  $\bar{R}_u: \bar{u}^\perp \rightarrow \bar{u}^\perp$  defined by  $\bar{R}_u(\bar{x}) = \pi(R_p(x, u)u)$ , for all  $\bar{x} = \pi(x) \in \bar{u}^\perp$ . It is easy to see that  $\bar{R}_u$  is a self-adjoint endomorphism, hence it is diagonalizable.

If  $z \in T_p M$  is a unit timelike vector, the *null congruence set* of  $z$  is defined to be the set  $N(z) = \{u \in T_p M \mid g_p(u, u) = 0, g_p(u, z) = -1\}$ . The elements of  $N(z)$  are in one-to-one correspondence to those of the set  $S(z) = \{x \in z^\perp \mid g_p(x, x) = 1\}$ , called the *celestial sphere of  $z$* , via the map  $\psi: N(z) \rightarrow S(z)$  such that  $\psi(u) = u - z$ .

**Definition 3.1** ([18, 20]). A Lorentzian manifold  $(M, g)$  is said to be *null Osserman with respect to  $z$* ,  $z \in T_p M$  being a unit timelike vector, if the eigenvalues of  $\bar{R}_u$  and their multiplicities are independent of  $u \in N(z)$ .

Following [7] and [8], we recall the basic facts related with the definition of the  $\varphi$ -null Osserman condition.

Let  $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$  be a Lorentzian *g.f.f.*-manifold, with  $\dim(M) = 2n + s$ , and  $\alpha \in \{1, \dots, s\}$ ,  $s \geq 1$ . It is easy to see that one of the characteristic vector fields has to be timelike and, without loss of generality, we assume it is  $\xi_1$ . If  $p \in M$ , we define the  *$\varphi$ -celestial sphere* of  $(\xi_1)_p$  to be the set  $S_\varphi((\xi_1)_p) = S((\xi_1)_p) \cap \text{Im}(\varphi_p)$ , and the  *$\varphi$ -null congruence set* of  $(\xi_1)_p$  to be  $N_\varphi((\xi_1)_p) = \psi^{-1}(S_\varphi((\xi_1)_p))$ .

**Definition 3.2** ([7, 8]). A Lorentzian *g.f.f.*-manifold  $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$  is said to be  *$\varphi$ -null Osserman with respect to  $(\xi_1)_p$* ,  $p \in M$ , if the eigenvalues of  $\bar{R}_u$  and their multiplicities are independent of  $u \in N_\varphi((\xi_1)_p)$ .

Fix  $p \in M$  and consider  $u \in N_\varphi((\xi_1)_p)$ . Since we can write  $u = (\xi_1)_p + x$ , with  $x \in S_\varphi((\xi_1)_p)$ , there is a natural one-to-one correspondence between the two kinds of Jacobi operator  $R_x: x^\perp \rightarrow x^\perp$  and  $\bar{R}_u: \bar{u}^\perp \rightarrow \bar{u}^\perp$ . In [8] it is provided the relationship between these two operators with respect to the  $\varphi$ -null Osserman condition, which we summarize in the following proposition.

**Proposition 3.3** ([8]). *Let  $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$  be a Lorentzian  $\mathcal{S}$ -manifold,  $\dim(M) = 2n + s$ ,  $s \geq 1$ . For any  $p \in M$ ,  $M$  is  $\varphi$ -null Osserman with respect to  $(\xi_1)_p$  if and only if the eigenvalues of  $R_x$  with their multiplicities are independent of  $x \in S_\varphi((\xi_1)_p)$ .*

The above result enables us to write the definition of the  $\varphi$ -null Osserman condition in terms of operator  $R_x$ ,  $x \in S_\varphi((\xi_1)_p)$ , instead of  $\bar{R}_u$ ,  $u \in N_\varphi((\xi_1)_p)$ . It is clear that, in the case of a Lorentzian Sasaki manifold, the  $\varphi$ -null Osserman condition reduces to that of null Osserman one.

### 4. PRINCIPAL TORUS BUNDLES AND THE $\varphi$ -NULL OSSERMAN CONDITION.

From [3] it is known that under an assumption of regularity it is possible to relate metric *g.f.f.*-manifolds both to almost complex and to almost contact metric manifolds via Riemannian submersions. The semi-Riemannian version of the results of [3] is provided in [10], where it is possible to find the following result.

**Theorem 4.1.** *Let  $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$  be a compact, connected and regular indefinite  $\mathcal{S}$ -manifold, with  $\dim(M) = 2n + s$ ,  $s \geq 2$ . Then, there exists a commutative diagram*

$$\begin{array}{ccc} M & \xrightarrow{\tau} & M' \\ & \searrow \pi \quad \swarrow \pi' & \\ & N & \end{array}$$

where  $N$  is a  $2n$ -dimensional compact Kähler manifold, either indefinite or not, and  $M'$  is a  $(2n + 1)$ -dimensional compact and regular Sasakian manifold, indefinite or not. All the maps are semi-Riemannian submersions with totally geodesic fibres, and more precisely:

- $\tau$  is the projection of a principal  $\mathbb{T}^{s-1}$ -bundle over  $M'$ ;
- $\pi'$  is the projection of a principal  $\mathbb{S}^1$ -bundle over  $N$ ;
- $\pi$  is the projection of a principal  $\mathbb{T}^s$ -bundle over  $N$ .

where  $\mathbb{T}^k$  is the  $k$ -dimensional torus, for any  $k \in \mathbb{N}$ ,  $k \geq 1$ .

For the notion of regularity of a distribution and of a  $g.f.f$ -structure the reader is referred to [34] and [3]. The general idea of this result, as contained in [3], is to fibrate  $M$  by any  $s - r$  of the vector fields  $\xi_\alpha$ 's, to obtain a principal  $\mathbb{T}^{s-r}$ -bundle over a  $(2n + r)$ -dimensional manifold  $M'$ . The remaining  $r$  characteristic vector fields are then projectable to  $M'$ , inducing a  $g.f.f$ -structure on  $M'$  and preserving the regularity. Thus,  $M'$  can be fibrated again by its  $r$  characteristic vector fields, obtaining a principal  $\mathbb{T}^r$ -bundle over  $N$ , which finally produces a commutative diagram. In particular, if we fibrate a Lorentzian  $\mathcal{S}$ -manifold  $M$  by the  $s - 1$  spacelike characteristic vector fields, in Theorem 4.1 we obtain that  $N$  is a Kähler manifold and  $M'$  is a Lorentz Sasakian manifold.

We are going to find out some informations about the possibility of projecting the  $\varphi$ -null Osserman condition both onto the null Osserman condition and the classical Osserman condition, via the previous fibrations.

In general (see [16], [31]), given a  $C^\infty$ -submersion  $f : (M, g) \rightarrow (B, g')$  between semi-Riemannian manifolds, i.e. a map whose differential  $(df)_p$  is surjective, for all  $p \in M$ , then  $\mathcal{V} = (\ker(df)_p)_{p \in M}$  and  $\mathcal{H} = (\ker(df)_p^\perp)_{p \in M}$  are, by definition, the *vertical* and the *horizontal* distributions of  $f$ . Such a map is said to be a *semi-Riemannian submersion* if each fibre  $f^{-1}(p')$ ,  $p' \in B$ , is a (semi-)Riemannian submanifold of  $M$  and the restriction of  $g_p$  to  $\mathcal{H}_p$  is an isometry for all  $p \in M$ . A vector field  $U$  (resp.  $X$ ) on  $M$  such that  $U_p \in \mathcal{V}_p$  (resp.  $X_p \in \mathcal{H}_p$ ) is called *vertical* (resp. *horizontal*). A vector field  $X$  on  $M$  such that there exists a vector field  $X'$  on  $B$  for which  $f_*X = X'$  is said to be *projectable*, and any horizontal, projectable vector field on  $M$  is said to be *basic*. The vertical distribution is always integrable, with the fibres of  $f$  as leaves. Denoting by  $v$  and  $h$  the projections of  $TM$  onto  $\mathcal{V}$  and  $\mathcal{H}$ , respectively, the *O'Neill tensors* of  $f$  are the  $(1, 2)$ -type tensor fields  $T$  and  $A$  on  $M$  defined by:

$$\begin{aligned} T(X, Y) &= T_X Y := v \nabla_{vX} hY + h \nabla_{vX} vY, \\ A(X, Y) &= A_X Y := v \nabla_{hX} hY + h \nabla_{hX} vY. \end{aligned}$$

They are both  $g$ -skew-symmetric tensors, and they satisfy the following fundamental properties:

$$\begin{aligned} T_U W &= T_W U & U, W &\in \mathcal{V} \\ A_X Y &= -A_Y X = \frac{1}{2} v[X, Y] & X, Y &\in \mathcal{H} \end{aligned}$$

It follows that the horizontal distribution is integrable if and only if  $A = 0$ , and in this case the leaves are totally geodesic submanifolds of  $M$ . Furthermore, the fibres of  $f$  are totally geodesic semi-Riemannian submanifolds of  $M$  if and only if  $T = 0$ .

**Lemma 4.2.** *Let  $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$  be a Lorentzian  $\mathcal{S}$ -manifold, with  $\dim(M) = 2n + s$ ,  $s \geq 1$ . Let  $\pi : M \rightarrow N$  be a principal  $\mathbb{T}^s$ -bundle over a Kähler manifold, as in Theorem 4.1. We have:*

$$(4.1) \quad A_X Y = -g(X, \varphi Y) \bar{\xi}, \quad A_X \xi_\alpha = -\varepsilon_\alpha \varphi X,$$

for any  $X, Y \in \text{Im}(\varphi)$  and any  $\alpha \in \{1, \dots, s\}$ , where  $\bar{\xi} = \sum_{\alpha=1}^s \xi_\alpha$ .

*Proof.* By construction of  $\pi$ , we have  $\mathcal{H}_p = \text{Im}(\varphi_p)$  and  $\mathcal{V}_p = \text{span}((\xi_1)_p, \dots, (\xi_s)_p)$  for any  $p \in M$ . Thus, since  $\nabla_X \xi_\alpha = -\varepsilon_\alpha \varphi X$ , by direct calculation we get:

$$\begin{aligned} A_X Y &= v(\nabla_X Y) = \sum_{\alpha=1}^s \varepsilon_\alpha g(\nabla_X Y, \xi_\alpha) \xi_\alpha = - \sum_{\alpha=1}^s \varepsilon_\alpha g(Y, \nabla_X \xi_\alpha) \xi_\alpha \\ &= \sum_{\alpha=1}^s g(Y, \varphi X) \xi_\alpha = -g(X, \varphi Y) \bar{\xi}, \end{aligned}$$

for all  $X, Y \in \mathcal{H}$ . Analogously, we have  $A_X \xi_\alpha = h(\nabla_X \xi_\alpha) = -\varepsilon_\alpha \varphi X$  for all  $X \in \mathcal{H}$  and  $\alpha \in \{1, \dots, s\}$ .  $\square$

For a semi-Riemannian submersion  $f : (M, g) \rightarrow (B, g')$ , let us denote by  $R^*$  the  $(1, 3)$ -type  $\mathcal{H}$ -valued tensor field on  $M$  such that, if  $X, Y, Z \in \Gamma(TM)$  are basic vector fields  $f$ -related to  $X', Y', Z' \in \Gamma(TB)$ , then  $R^*(X, Y)Z$  is the unique basic vector field  $f$ -related to  $R'(X', Y')Z'$ . Thus, for any  $x \in \mathcal{H}_p$ , one can consider the self-adjoint endomorphism  $R_x^* : x^\perp \cap \mathcal{H}_p \rightarrow x^\perp \cap \mathcal{H}_p$  such that  $R_x^*(y) = R_p^*(y, x)x$ .

**Lemma 4.3.** *Let  $f : (M, g) \rightarrow (B, g')$  be a semi-Riemannian submersion. For any orthogonal vectors  $x, y \in \mathcal{H}_p$  one has*

$$(4.2) \quad R_x^*(y) = h_p R_x(y) - 3A_x A_x(y).$$

*Proof.* From standard formulas on the curvature tensors of a submersion (see [16], pag. 13), we have

$$\begin{aligned} g_p(R_x^*(y), z) &= R_p^*(x, y, x, z) \\ &= R_p(x, y, x, z) + 2g_p(A_x(y), A_x(z)) - g_p(A_y(x), A_x(z)) \\ &= g_p(h_p R_x(y), z) - 3g_p(A_x A_x(y), z) \end{aligned}$$

for any  $z \in x^\perp \cap \mathcal{H}_p$ , which yields (4.2).  $\square$

**Proposition 4.4.** *Let  $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$  be a Lorentzian  $\mathcal{S}$ -manifold, with  $\dim(M) = 2n + s$ ,  $s \geq 1$ . Let  $\pi : M \rightarrow N$  be a principal  $\mathbb{T}^s$ -bundle over a Kähler manifold  $(N, J, G)$ , as in Theorem 4.1. Let  $p \in M$ , and suppose that, for any  $x \in S_\varphi((\xi_1)_p)$ ,  $\varphi x$  is an eigenvector of  $R_x$ . Then,  $M$  is  $\varphi$ -null Osserman with respect to  $(\xi_1)_p$  if and only if  $N$  is Osserman at  $p' = \pi(p)$ .*

*Proof.* Suppose first that  $s \geq 2$ . Fix  $p' \in N$ , with  $p' = \pi(p)$ ,  $p \in M$ , and let  $x' \in T_{p'} N$  a unit vector, and  $y', z' \in x'^\perp$ . Let  $x \in S_\varphi((\xi_1)_p)$ ,  $V = x^\perp \cap \text{Im}(\varphi_p)$  and  $y, z \in V$  such that  $x' = (d\pi)_p(x)$ ,  $y' = (d\pi)_p(y)$  and  $z' = (d\pi)_p(z)$ . Then

$$g_p(R_x^*(y), z) = G_{p'}((d\pi)_p(R_x^*(y)), (d\pi)_p(z)) = G_{p'}(R'_{x'}(y'), z'),$$

which implies that the Jacobi operators  $R_x^* : V \rightarrow V$  and  $R'_{x'} : x'^\perp \rightarrow x'^\perp$  have the same characteristic polynomial. Using (4.1) one has  $A_x A_x(y) = -(s-2)g_p(y, \varphi x)\varphi x$  and since  $R_x$  leaves the subspace  $V$  invariant, (4.2) gives

$$R_x^*(y) = R_x(y) + 3(s-2)g_p(y, \varphi x)\varphi x$$

for any  $y \in V$ . Observe that if  $\varphi x$  is an eigenvector of  $R_x$ , we have

$$g_p(R_x(y), \varphi x)\varphi x = g_p(y, R_x(\varphi x))\varphi x = R_x(g_p(y, \varphi x)\varphi x),$$

that is the endomorphism of  $V$  such that  $y \mapsto g_p(y, \varphi x)\varphi x$  commutes with  $R_x$ . This implies they are simultaneously diagonalizable, and if  $\lambda_i$ ,  $i \in \{1, \dots, r\}$  are the eigenvalues of  $R_x$ , counted with multiplicities, with  $\lambda_1$  relative to  $\varphi x$ , then  $\lambda_1 + 3(s-2)$ ,  $\lambda_j$ ,  $j \in \{2, \dots, r\}$  are the eigenvalues of  $R_x^*$ . By Proposition 3.3 we obtain our statement.

If  $s = 1$  then the proof goes through as above, except for the fact that one has  $A_x A_x(y) = g_p(y, \varphi x)\varphi x$ .  $\square$

**Remark 4.5.** It is clear, from the previous proof, that in case  $s = 2$  the hypothesis of  $\varphi x$  being an eigenvector of  $R_x$  can be dropped without affecting the result. Furthermore, in case  $s = 1$ , the statement is relative to the projection  $\pi'$  of the commutative diagram in the Theorem 4.1.

**Lemma 4.6.** *Let  $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$  be a Lorentzian  $\mathcal{S}$ -manifold, with  $\dim(M) = 2n + s$ ,  $s \geq 2$ . Let  $\tau: M \rightarrow M'$  be a principal  $\mathbb{T}^{s-1}$ -bundle over a Lorentz Sasakian manifold, as in Theorem 4.1. We have:*

$$(4.3) \quad A_X Y = -g(X, \varphi Y) \sum_{\alpha=2}^s \xi_\alpha, \quad A_X \xi_\alpha = -\varphi X,$$

for any  $X, Y \in \text{Im}(\varphi) \oplus \text{span}(\xi_1)$  and any  $\alpha \in \{2, \dots, s\}$ .

*Proof.* By construction of  $\tau$ , we have the splitting  $\mathcal{H}_p = \text{Im}(\varphi_p) \oplus \text{span}(\xi_1)$  and  $\mathcal{V}_p = \text{span}((\xi_2)_p, \dots, (\xi_s)_p)$  for any  $p \in M$ . Proceeding along the same lines as the proof of Lemma 4.2, we get (4.3).  $\square$

**Proposition 4.7.** *Let  $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$  be a Lorentzian  $\mathcal{S}$ -manifold, with  $\dim(M) = 2n + s$ ,  $s \geq 2$ . Let  $\tau: M \rightarrow M'$  be a principal  $\mathbb{T}^{s-1}$ -bundle over a Lorentz Sasakian manifold  $M'$  with structure  $(\varphi', \xi', \eta', g')$  as in Theorem 4.1. Let  $p \in M$ , and suppose that, for any  $x \in S_\varphi((\xi_1)_p)$ ,  $\varphi x$  is an eigenvector of  $R_x$ . Then,  $M$  is  $\varphi$ -null Osserman with respect to  $(\xi_1)_p$  if and only if  $M'$  is null Osserman with respect to  $\xi'_{p'}$ ,  $p' = \tau(p)$ .*

*Proof.* One can follow the same proof of Proposition 4.4 where, using (4.3), one has  $A_x A_x(y) = -(s-1)g_p(y, \varphi x)\varphi x$ .  $\square$

Propositions 4.4 and 4.7 can be summarized as follows.

**Theorem 4.8.** *Let  $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$  be a compact, connected and regular Lorentzian  $\mathcal{S}$ -manifold, with  $\dim(M) = 2n + s$ ,  $s \geq 2$ . Consider the commutative diagram of principal torus bundles*

$$\begin{array}{ccc} M & \xrightarrow{\tau} & M' \\ & \searrow \pi & \swarrow \pi' \\ & N & \end{array}$$

where  $N$  is a  $2n$ -dimensional compact Kähler manifold and  $M'$  is a  $(2n+1)$ -dimensional compact and regular Lorentz Sasakian manifold, with unit timelike characteristic vector field  $\xi' = \tau_*(\xi_1)$ . Let  $p \in M$ , and suppose that  $\varphi x$  is an eigenvector of  $R_x$  for any  $x \in S_\varphi((\xi_1)_p)$ . The following three statements are equivalent.

- (a)  $M$  is  $\varphi$ -null Osserman with respect to  $(\xi_1)_p$ ;
- (b)  $N$  is Osserman at  $q = \pi(p)$ ;
- (c)  $M'$  is null Osserman with respect to  $\xi'_{p'}$ ,  $p' = \tau(p)$ .

**Remark 4.9.** It is clear that the three Osserman-type conditions in the above theorem can be also considered either pointwise or globally. Moreover, if we use

the pointwise conditions, from the equivalence (a)  $\Leftrightarrow$  (b) it follows that  $N$  is Einstein at each point and the connectedness implies that it is a Kähler-Einstein manifold.

**Remark 4.10.** In case  $\tau : M \rightarrow M'$  is a principal  $\mathbb{T}^{s-1}$ -bundle from a Lorentzian  $\mathcal{S}$ -manifold  $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$  with  $\dim(M) = 2n + s$ ,  $s \geq 2$ , over a Sasakian manifold  $M'$  with structure  $(\varphi', \xi', \eta', g')$  as in Theorem 4.1, we could ask about the Osserman condition on  $M'$ . Let us suppose  $M'$  pointwise Osserman, since it is odd-dimensional, it has constant sectional curvature  $c$  ([12, 20]). Being  $k(X', \xi') = 1$ , for any  $X' \in \text{Im}(\varphi')$ , then  $c = 1$  and  $M'$  is locally isometric to the sphere  $\mathbb{S}^{2n+1}$  with its standard Sasakian structure (see [2], p. 114). By construction of the bundle projection  $\tau$ , we can suppose that  $\mathcal{H}_p = \text{Im}(\varphi_p) \oplus \text{span}((\xi_s)_p)$  and  $\mathcal{V}_p = \text{span}((\xi_1)_p, \dots, (\xi_{s-1})_p)$ . Hence, with calculations similar to those of Lemma 4.2, one has  $A_X Y = g(Y, \varphi X) \sum_{\alpha=1}^{s-1} \xi_\alpha$ . By standard formulas on sectional curvatures of the total and the base spaces of a semi-Riemannian submersion (see [16], p. 14) we have

$$k(x, \varphi x) = k'(x', \varphi' x') - 3g(A_x \varphi x, A_x \varphi x) = 1 - 3(s - 3), \quad x \in \text{Im}(\varphi_p),$$

which gives a necessary condition on the  $\varphi$ -sectional curvature of  $M$  for  $M'$  to be an Osserman Sasakian manifold.

**Remark 4.11.** Analogously, in case  $\tau : M \rightarrow M'$  is a principal  $\mathbb{T}^{s-1}$ -bundle from a Lorentzian  $\mathcal{S}$ -manifold  $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ , with  $\dim(M) = 2n + s$ ,  $s \geq 2$ , over a Lorentz Sasakian manifold  $M'$  with structure  $(\varphi', \xi', \eta', g')$  as in Theorem 4.1, we could ask again about the Osserman condition on  $M'$ . It is known that any connected Lorentzian Osserman manifold is a space-form ([20]), and since  $k(X', \xi') = -1$ , for any  $X' \in \text{Im}(\varphi')$ ,  $M'$  has constant sectional curvature  $c = -1$ . As in the previous calculations, using (4.3), we have

$$k(x, \varphi x) = k'(x', \varphi' x') - 3g(A_x \varphi x, A_x \varphi x) = -1 - 3(s - 1), \quad x \in \text{Im}(\varphi_p),$$

which is a necessary condition on the  $\varphi$ -sectional curvature of  $M$  for  $M'$  to be a Lorentzian Osserman manifold.

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